ON THE MODEL OF A SKEW–SELFADJOINT OPERATOR WITH A SIMPLE SPECTRUM ON A HILBERT QUATERNION MODULE

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Abstract. In this work we construct the model of a skew–selfadjoint operator with a simple spectrum acting on a Hilbert quaternion bimodule. This result is based on the Spectral Theorem for a skew–selfadjoint operator. In the case of a bounded normal operator this Spectral Theorem was announced in the report [3]. The more detailed research was carried out in the paper [4].

In the same paper we pointed out that the given reasonings enable us to prove corresponding results for a unbounded skew–selfadjoint operators too. Results of this article essentially develop the results[1] of the paper [1].

Keywords. Quaternion, Hilbert space, quaternion module, quaternion bimodule, spectral theorem, normal operator, skew–selfadjoint operator, operator with a simple spectrum, model of a linear operator, operator of left multiplication by an independent variable.

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Introduction

On the skew field of quaternions $\mathbb{H}$ and $\mathbb{R}$–containing subfields of $\mathbb{H}$. For reader's convenience recall some definitions and facts concerning quaternions.

The real quaternionic skew-field $\mathbb{H}$ is a four-dimensional associative division algebra of a rank 4 over $\mathbb{R}$ with the basis $\{1, i, j, k\}$ and multiplication rules

$$\begin{array}{lll} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

For any $q \in \mathbb{H}$ there exist unique q_0, q_1, q_2, $q_3 \in \mathbb{R}$ such that $q = q_0 + q_1 i + q_2 j + q_3 k$ (*real representation* of q). Also it is useful to deal with *the vector form* of quaternion $q = q_0 + \vec{q}$ where $\vec{q} = q_1 i + q_2 j + q_3 k$ is the *vector* or *imaginary part* of q (if $q = \vec{q}$ then q is called to be a *vector* or an *imaginary* quaternion).

For instance the vector form for the product of quaternions q and p is nothing more than the well-known formula of multiplication $qp = q_0 p_0 - (\vec{q}, \vec{p}) + \big([\vec{q}, \vec{p}] + p_0 \vec{q} + q_0 \vec{p}\big)$. Here

[1]Note that since the issue of [1] investigations of spectral problems in *infinite–dimensional* Hilbert (bi)modules were not carried out as far as we know. As it seems for us our papers [4, 5] are the first in this row after [1].

$(\cdot,\cdot)$ is the usual scalar product and $[\cdot,\cdot]$ is the vector product on the three–dimensional space $\mathbb{R}\langle i,j,k\rangle$ of vector quaternions. *Conjugate* of q is defined by $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$. The map $q \to \overline{q}$ is an involution on $\mathbb{H}$, and $q\overline{q} = \overline{q}q = \sum_{t=0}^{3} q_t^2 \in \mathbb{R}$. One can define the *absolute value* $|q|$ of q by $|q| = (q\overline{q})^{1/2}$. Thus, we can consider $\mathbb{H}$ as a *normed algebra*. An imaginary quaternion whose absolute value equals 1 is called an *imaginary unit*. From this point of view one can consider the set of complex numbers $\mathbb{C}$ as a real subalgebra of $\mathbb{H}$: $\mathbb{C} = \mathbb{R}\langle 1,i\rangle$. Inclusion $\mathbb{C}$ in $\mathbb{H}$ allows us to obtain the *complex* (or *symplectic*) representation of a quaternion. Namely, for $q = q_0 + q_1 i + q_2 j + q_3 k$ we have $q = (q_0 + q_1 i) + (q_2 + q_3 i)j = z_1 + z_2 j$ where $z_1,\ z_2 \in \mathbb{C}$.

Often it is useful to restrict not oneself by choice of a concrete field; e.g. $\mathbb{C}$. In fact $\mathbb{C}$ is just a specimen of a field in $\mathbb{H}$, which extends $\mathbb{R}$. Everywhere in this paper we denote such a field by $\mathbb{F}$. $\mathbb{F}$ is a commutative and associative division algebra over $\mathbb{R}$; and dimension of $\mathbb{F}$ is greater than 1. Hence by Frobenius theorem dimension of $\mathbb{F}$ equals 2; and $\mathbb{F}$ is isomorphic to $\mathbb{C}$.

A field $\mathbb{F}$ is uniquely determined by some nonreal quaternion. Indeed, let $q \in \mathbb{F} \setminus \mathbb{R}$. Write down this quaternion in the vector form: $q = q_0 + \vec{q}$. Then we have $\vec{q} = q - q_0 \in \mathbb{F} \setminus \{0\}$. Let $f := \frac{1}{|\vec{q}|}\,\vec{q}$. The vector system $\{1, f\}$ is linear independent. Hence it is a basis of $\mathbb{F}$ as a two–dimensional $\mathbb{R}$–algebra. These arguments show that any two nonreal elements of $\mathbb{F}$ have proportional vector parts, and this condition is necessary and sufficient for quaternions with the corresponding vector parts to commute. Thus, we can characterize any subfield of $\mathbb{H}$ satisfying the above conditions as a set of all quaternions commuting with some fixed nonreal quaternion. In addition this necessary and sufficient condition implies possibility for a corresponding imaginary unit f to be determined by $\mathbb{F}$ up to ± 1.

Consider now $\mathbb{H}$ as a real Euclidean space with dimension 4. Choose any normed quaternion ϕ to be orthogonal to $1, f$. Then we have $\phi^2 = (\phi,\phi) = -1$. The quaternion $f\phi\,(= [f,\phi])$ is orthogonal to $1, f, \phi$; and $f\phi = [f,\phi] = -[\phi, f] = -\phi f$. Hence $(f\phi)^2 = -1$. Thus, the system $\{1, f, \phi, f\phi\}$ is an $\mathbb{R}$–basis of $\mathbb{H}$; and this basis consists of 1 and three imaginary units. This fact allows us to define the unique decomposition $q = q_0 + \widetilde{q}_1 f + \widetilde{q}_2\phi + \widetilde{q}_3 f\phi$ which implies an analogue of a complex decomposition $q = (q_0 + \widetilde{q}_1 f) + (\widetilde{q}_2 + \widetilde{q}_3 f)\phi = u_1 + u_2\phi$ $(u_1,\ u_2 \in \mathbb{F})$. Note that a corresponding quaternion ϕ for a field $\mathbb{F}$ is not uniquely determined.

Agreements and notations. In the paper we use substantially accepted notations. Some notations we define as one goes along (e.g. Lin, Clos, p. 4).

"Linear operator" in $\mathbb{H}$–bimodule means a *right–side* linear operator. (All the results presented here hold true if term "bimodule" one change by term "right module". Consideration of precisely bimodules is connected with the tradition of our previous papers).

An operator of right multiplication $R_q x := xq$ $(x \in H)$, $q \in \mathbb{H}$, plays a significant role in studying operators on quaternion bimodules (R_q is not $\mathbb{H}$–linear but it is $\mathbb{F}$–linear

where the field $\mathbb{F}$ is generated by q; see above). It is useful to emphasize the following simple properties of an operator of right multiplication which we use throughout the paper without any special comments: $R_{p+q} = R_p + R_q$; $R_{pq} = R_q R_p$.

Note that an $\mathbb{R}$–*linear operator* A is $\mathbb{F}$–*linear* iff $R_f A = A R_f$ (f is an imaginary unit generating $\mathbb{F}$). $\mathbb{F}$–*linear operator* A is $\mathbb{H}$–*linear* iff $R_\phi A = A R_\phi$.

For further consideration we fix an imaginary unit f and a field $\mathbb{F}$ generated by f. In conclusion note that all arguments concerning $\mathbb{F}$–modules (orthogonality; etc.) copy the corresponding ones for $\mathbb{C}$–modules because of isometrical isomorphism[2] between $\mathbb{F}$ and $\mathbb{C}$. In this case we use the corresponding notations $\perp_{\mathbb{F}}$, $\oplus_{\mathbb{F}}$; etc.

The model of a skew–selfadjoint operator with a simple spectrum

Spectral Theorem. The spectral theorem for a skew–selfadjoint operator mentioned in introduction can be written as follows.

Theorem 1. *Let A be a skew–selfadjoint operator acting on a Hilbert quaternion bimodule H (whose domain $\mathcal{D}(A)$ is dense in H), $\mathbb{F}$ be an $\mathbb{R}$–containing subfield of $\mathbb{H}$. Then there exists a spectral measure E defined on the Borel σ–algebra $\mathfrak{B}(\mathbf{f}_+)$ (consisting of all Borel subsets of the half–axis $\mathbf{f}_+ = \{\tau f \mid \tau \geqslant 0\}$), skew–selfadjoint operator J commuting with E and satisfying the condition $J^2 = -I$ such that the following equalities hold*

$$\mathcal{D}(A) = \{h \in H \mid \int\limits_{\sigma_{\mathbb{F}_+}(A)} |q|^2 \langle E(dq)h, h\rangle < \infty\}\,; \tag{1}$$

$$A = \int\limits_{\mathbf{f}_+} R_{-qf} J E(dq)\,. \tag{2}$$

The integral in (2) is understood as a strong limit

$$Ah = \lim_{n\to\infty} \int\limits_{[0,nf]} R_{-qf} J E(dq) h\,, \quad h \in \mathcal{D}(A).$$

We can determine

$$E(\alpha) := \begin{cases} E_{\mathbb{F}}(\alpha) + E_{\mathbb{F}}(-\alpha),\ 0 \notin \alpha, \\ E_{\mathbb{F}}(\alpha) + E_{\mathbb{F}}(-\alpha) - E_{\mathbb{F}}(\{0\}),\ 0 \in \alpha \end{cases} \qquad \big(\alpha \in \mathfrak{B}(\mathbf{f}_+)\big) \tag{3}$$

where $E_{\mathbb{F}}$ is a $\mathbb{F}$–linear spectral measure defined on Borel subsets of the axis $\mathbf{f} = f\mathbb{R}$; i.e. the spectral measure of A where A acts on H as on a $\mathbb{F}$–module. The last is isometrically

[2]We pay reader's attention to the following remarkable phenomenon arising in the quaternion theory of functions and operators. This phenomenon is connected with the relation between such notions as "equality" and "isomorphism". Although every such a field $\mathbb{F}$ is isomorphic to $\mathbb{C}$ and therefore it is "not interesting" from the *algebraic* point of view nevertheless it is a *unique* "copy" of $\mathbb{C}$ in $\mathbb{H}$. This fact becomes crucial for *analysis*. In particular our theory of differentiability of functions of a quaternionic variable ([2]) is based precisely on "playing by" these properties.

isomorphic to H as to a $\mathbb{C}$–module, therefore, we can use the classical spectral theorem for constructing $E_{\mathbb{F}}$. In addition, J and $E_{\mathbb{F}}$ are connected by the formula

$$J = R_f\big(E_{\mathbb{F}}(\mathbf{f}_+) - E_{\mathbb{F}}(\mathbf{f}_-)\big) \tag{4}$$

where $\mathbf{f}_- = \{\tau f \mid \tau < 0\}$ is the negative half–axis of $\mathbb{F}$. Concerning (3) note that $\mathbb{H}$–linearity of E is equivalent to validity of the equality

$$E_{\mathbb{F}}(\alpha)R_\phi = R_\phi E_{\mathbb{F}}(-\alpha) \quad \big(\alpha \in \mathfrak{B}(\mathbf{f}_+)\big). \tag{5}$$

In particular, putting $H^+ = E_{\mathbb{F}}(\mathbf{f}_+)H$ we have the decomposition

$$H = H^+ \oplus_{\mathbb{F}} R_\phi H^+. \tag{6}$$

A skew–selfadjoint operator with a simple spectrum. Let A be a skew–selfadjoint operator acting on a Hilbert quaternion bimodule H. By theorem 1 every such an operator determines the spectral measure E and the corresponding operator J commuting with E.

Denote by $\mathcal{I}$ the set of all (bounded) intervals Δ of half-axis $\mathbf{f}_+$; $\Delta = [af, bf]$; $a, b \in \mathbb{R}$; $0 \leqslant a < b$.

Definition 1. A is called to be an operator *with a simple spectrum* if there exists a *generating* vector $g \in H$:

$$\operatorname{Clos}\operatorname{Lin}\{E(\Delta)g \mid \Delta \in \mathcal{I}\} = H \tag{7}$$

(Lin is the (right-side) $\mathbb{H}$–span, Clos is the H–norm closure).

A special generating vector. Now we prove the key fact for further consideration of existence of a generating vector g with the property

$$Jg = R_f g. \tag{8}$$

Let $\mathcal{T} = \{T_i\}_{i\in I}$ be a set of operators on H, $g \in H$ is a vector. Define

$$\mathcal{C}(\mathcal{T}, g) = \operatorname{Clos}\operatorname{Lin}\{T_i g \mid i \in I\}; \quad \mathcal{C}_{\mathbb{F}}(\mathcal{T}, g) = \operatorname{Clos}\operatorname{Lin}_{\mathbb{F}}\{T_i g \mid i \in I\}$$

($\operatorname{Lin}_{\mathbb{F}}$ is the (right–side) $\mathbb{F}$–span). The following properties are quite elementary:

$$\mathcal{C}_{\mathbb{F}}(\mathcal{T}, g_1 + g_2) \subseteq \operatorname{Clos}\big(\mathcal{C}_{\mathbb{F}}(\mathcal{T}, g_1) + \mathcal{C}_{\mathbb{F}}(\mathcal{T}, g_2)\big); \tag{9}$$

$$\mathcal{C}(\mathcal{T}, g) = \operatorname{Clos}\big(\mathcal{C}_{\mathbb{F}}(\mathcal{T}, g) + R_\phi \mathcal{C}_{\mathbb{F}}(\mathcal{T}, g)\big). \tag{10}$$

If the set $\mathcal{T}$ is closed respectively to the product of operators then we have

$$h \in \mathcal{C}_{\mathbb{F}}(\mathcal{T}, g) \Rightarrow \mathcal{C}_{\mathbb{F}}(\mathcal{T}, h) \subseteq \mathcal{C}_{\mathbb{F}}(\mathcal{T}, g) \quad (if \ \ \forall\, i, j \in I \quad T_i T_j \in \mathcal{T}) \tag{11}$$

(the same is true for $\mathcal{C}(\cdot,\cdot)$). Denote

$$\mathcal{E} = \{E(\Delta) \mid \Delta \in \mathcal{J}\}, \quad \mathcal{E}_{\mathbb{F}} = \{E_{\mathbb{F}}(\Delta) \mid \Delta \in \mathcal{J}\}.$$

By use of (9) — (11) the following relations can be easily derived from the properties of a spectral measure:

$$\forall\, g \in H \quad \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, g) \subseteq H^+; \tag{12}$$

$$g \in H^+ \Rightarrow \mathcal{C}(\mathcal{E}, g) = \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, g); \tag{13}$$

$$\forall\, g \in H \quad \mathcal{C}(\mathcal{E}, R_\phi g) = R_\phi \mathcal{C}(\mathcal{E}, g); \tag{14}$$

$$h \perp_{\mathbb{F}} \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, g) \Rightarrow \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, h+g) = \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, h) \oplus_{\mathbb{F}} \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, g). \tag{15}$$

Statement 1. *Let A be an operator with a simple spectrum. Then there exists*[3] *a generating vector g satisfying* (8).

Proof. Let y be an arbitrary generating vector for A. Then we have

$$H = \mathcal{C}(\mathcal{E}, y). \tag{16}$$

By (6) for some vectors $y^+, x^+ \in H^+ \quad y = y^+ + R_\phi x^+$. Denote

$$\mathfrak{Y}^+ = \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, y^+), \quad \mathfrak{X}^+ = \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, x^+). \tag{17}$$

Then we have

$$\mathcal{C}_{\mathbb{F}}(\mathcal{E}, y) \overset{(9)}{\subseteq} \operatorname{Clos}\big(\mathcal{C}(\mathcal{E}, y^+) + \mathcal{C}(\mathcal{E}, R_\phi x^+)\big) \overset{(14),(13),(17)}{=} \operatorname{Clos}(\mathfrak{Y}^+ + R_\phi \mathfrak{X}^+). \tag{18}$$

Hence

$$R_\phi \mathcal{C}_{\mathbb{F}}(\mathcal{E}, y) \subseteq \operatorname{Clos}(R_\phi \mathfrak{Y}^+ + \mathfrak{X}^+). \tag{19}$$

Furthermore,

$$\begin{aligned} H &\overset{(16),(10),(18),(19)}{\subseteq} \operatorname{Clos}\big(\mathfrak{Y}^+ + R_\phi \mathfrak{X}^+ + R_\phi \mathfrak{Y}^+ + \mathfrak{X}^+\big) \overset{(6),(12)}{=} \\ &= \operatorname{Clos}\big((\mathfrak{Y}^+ + \mathfrak{X}^+) \oplus_{\mathbb{F}} R_\phi(\mathfrak{Y}^+ + \mathfrak{X}^+)\big) = \operatorname{Clos}(\mathfrak{Y}^+ + \mathfrak{X}^+) \oplus_{\mathbb{F}} R_\phi \operatorname{Clos}(\mathfrak{Y}^+ + \mathfrak{X}^+); \end{aligned} \tag{20}$$

then by (6), (12)

$$H^+ = \operatorname{Clos}(\mathfrak{Y}^+ + \mathfrak{X}^+). \tag{21}$$

Let v^+ be an $\mathbb{F}$–orthogonal projection of the vector x^+ on $H^+ \ominus_{\mathbb{F}} \mathfrak{X}^+$; and $x^+ = v^+ + w^+,\ w^+ \in \mathfrak{X}^+$. Let $g = y^+ + v^+$. Since $g \in H^+$; therefore, by (4) g satisfies (8). Now we can easily prove that g is a generating vector. Denote $\mathfrak{V}^+ = \mathcal{C}_{\mathbb{F}}(\mathcal{E}, v^+)$. By (11)

$$\mathcal{C}_{\mathbb{F}}(\mathcal{E}, w^+) \subseteq \mathfrak{Y}^+. \tag{22}$$

Hence

$$\mathfrak{Y}^+ + \mathfrak{X}^+ = \mathfrak{Y}^+ + \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, v^+ + w^+) \overset{(15)}{=} \mathfrak{Y}^+ + \mathfrak{V}^+ + \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, w^+) \overset{(22),(15)}{=} \mathfrak{Y}^+ \oplus_{\mathbb{F}} \mathfrak{V}^+. \tag{23}$$

On the other hand

$$H^+ \overset{(21),(23)}{=} \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, y^+) \oplus_{\mathbb{F}} \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, v^+) \overset{(15)}{=} \mathcal{C}_{\mathbb{F}}(\mathcal{E}_{\mathbb{F}}, y^+ + v^+) \overset{(13)}{=} \mathcal{C}_{\mathbb{F}}(\mathcal{E}, g). \tag{24}$$

By (24), (10), (6) we obtain $H = \mathcal{C}(\mathcal{E}, g)$; i.e. g is a generating vector. □

[3] The main idea how to construct a special generating vector in statement 1 is borrowed from the [1, Prop. 5.2].

Model. Further we assume that a generating vector g of a skew–selfadjoint operator A with a simple spectrum satisfies (16).

Consider the operator

$$(Qh)(\lambda) = \lambda h(\lambda)$$

acting on the $\mathbb{H}$–bimodule $L^2_\sigma(\mathbf{f}_+, \mathbb{H})$ of all square integrable by a measure σ $\mathbb{H}$–valued functions on the half–axis $\mathbf{f}_+$. Since the domain of Q is

$$\mathcal{D}(Q) = \big\{ h \in L^2_\sigma \mid \int\limits_{\mathbf{f}_+} |\lambda|^2 \, |h(\lambda)|^2 \sigma(d\lambda) < \infty \big\};$$

therefore, by standard arguments one can show that Q is a skew–selfadjoint operator.

To find out the spectral measure $E_{\mathbb{F}}$ we use a slight modification of the standard algorithm which was considered in our paper [4]. In particular, for any interval (af, bf), $-\infty < a < b < \infty$; and any function h from $\mathcal{D}(Q)$ which has the form $h(\lambda) = h_1(\lambda) + h_2(\lambda)\phi$ where $h_1(\lambda), h_2(\lambda) \in \mathbb{F}$ we have

$$\begin{aligned} E_{\mathbb{F}}\big((af, bf)\big)h(\lambda) &= \chi_{(a,b)}(-\lambda f)h_1(\lambda) + \chi_{(a,b)}(\lambda f)h_2(\lambda)\phi = \\ &= \chi_{(af,bf)}(\lambda)h_1(\lambda) + \chi_{(-bf,-af)}(\lambda)h_2(\lambda)\phi. \end{aligned}$$

Then for any $\alpha \in \mathfrak{B}(\mathbf{f})$

$$E_{\mathbb{F}}(\alpha)h(\lambda) = \chi_\alpha(\lambda)h_1(\lambda) + \chi_{-\alpha}(\lambda)h_2(\lambda)\phi.$$

Hence for $\alpha \in \mathfrak{B}(\mathbf{f}_+ \backslash \{0\})$

$$E_{\mathbb{F}}(\alpha)h(\lambda) = \chi_\alpha(\lambda)h_1(\lambda), \;\; E_{\mathbb{F}}(-\alpha)h(\lambda) = \chi_\alpha(\lambda)h_2(\lambda)\phi;$$

and, finally,

$$E(\alpha)h(\lambda) = \chi_\alpha(\lambda)h(\lambda).$$

If $\alpha \in \mathfrak{B}(\mathbf{f}_+), \; 0 \in \alpha$, then

$$\begin{cases} E_{\mathbb{F}}(\alpha)h(\lambda) & = \chi_\alpha(\lambda)h_1(\lambda) + \widetilde{h}_2(\lambda)\phi, \\ E_{\mathbb{F}}(-\alpha)h(\lambda) & = \widetilde{h}_1(\lambda) + \chi_\alpha(\lambda)h_2(\lambda)\phi \end{cases}$$

where

$$\widetilde{h}_r(\lambda) = \begin{cases} h_r(0), & \lambda = 0, \\ 0, & \lambda \neq 0, \end{cases} \qquad (r = 1, 2).$$

By the equality

$$E_{\mathbb{F}}(\{0\})h(\lambda) = \begin{cases} h(0), & \lambda = 0, \\ 0, & \lambda \neq 0 \end{cases}$$

one can also obtain the equality

$$E(\alpha)h(\lambda) = \chi_\alpha(\lambda)h(\lambda).$$

By virtue of (4) and above formulae the corresponding operator J for Q has the form

$$(Jh)(\lambda) = (h_1(\lambda) - h_2(\lambda)\phi)f.$$

(almost everywhere and except 0)

Define the step function

$$g(\lambda) = \alpha_k,\ \lambda \in \Delta_k = \big((k-1)f, kf\big),\ \alpha_k \in \mathbf{f}_+ \backslash \{0\},$$

with the condition $\sum |\alpha_k|^2 \sigma(\Delta_k) < \infty$. This function generates Q in the sense of definition 1. Indeed, the (right) quaternionic span of the set of functions $E(\Delta)g$ coincides with the set of all finite step functions and this set is dense in $L^2_\sigma(\mathbf{f}_+, \mathbb{H})$. Note that $(Jg)(\lambda) = \alpha_k f,\ \lambda \in \Delta_k$; i.e. $Jg = R_f g$. Briefly denote $L^2_\sigma(\mathbf{f}_+, \mathbb{H})$ by L^2_σ.

Theorem 2. *Let A be a skew–selfadjoint operator with a simple spectrum acting on a Hilbert quaternion bimodule H, g be a generating vector, $\sigma(\alpha) = \langle E(\alpha)g, g\rangle$ be a measure defined on the σ–algebra $\mathfrak{B}(\mathbf{f}_+)$. Then the map $\Phi : L^2_\sigma \to H$ determined by the integral*

$$\Phi h = \int_{\mathbf{f}_+} R_{h(\lambda)} E(d\lambda) g,$$

sets an isometric isomorphism between L^2_σ and H such that

$$A\Phi h = \int_{\mathbf{f}_+} R_{\lambda h(\lambda)} E(d\lambda) g.$$

Proof. Denote

$$G = \Big\{\widehat{h} \in H \mid \exists\, h \in L^2_\sigma :\ \widehat{h} = \int_{\mathbf{f}_+} R_{h(\lambda)} E(d\lambda) g\Big\} = \Im(\Phi)$$

(range of Φ). Let Δ be an interval of $\mathbf{f}_+$. Then $\chi_\Delta \in L^2_\sigma$; and

$$\Phi \chi_\Delta = \int_{\mathbf{f}_+} E(d\lambda) g \chi_\Delta(\lambda) = \int_{\Delta} E(d\lambda) g = E(\Delta) g.$$

Hence G is dense in H. Let $\widehat{h} \in G$. Then $\langle \widehat{h}, \widehat{h}\rangle_H = \int_{\mathbf{f}_+} \langle E(d\lambda) g, \widehat{h}\rangle h(\lambda)$. Since

$$\langle E(\alpha) g, \widehat{h}\rangle = \int_{\mathbf{f}_+} \langle E(\alpha) g, E(d\lambda) g h(\lambda)\rangle = \int_{\mathbf{f}_+} \overline{h(\lambda)} \langle E(d\lambda) E(\alpha) g, g\rangle =$$

$$= \int_{\alpha} \overline{h(\lambda)} \langle E(d\lambda) g, g\rangle = \int_{\alpha} \overline{h(\lambda)} \sigma(d\lambda);$$

therefore,

$$\langle \widehat{h}, \widehat{h}\rangle_H = \int_{\mathbf{f}_+} |h(\lambda)|^2 \sigma(d\lambda) = \langle h(\lambda), h(\lambda)\rangle_{L^2_\sigma}.$$

Thus, Φ isometrically maps the dense subset of L^2_σ onto a dense subset of G. Density of L^2_σ yields closedness of G. Hence $G = H$; and Φ is an unitary operator.

Next prove the second part of the theorem. Let h be a finite function from L^2_σ with a support $[af, bf]$; $\widehat{h} = \int_{\mathbf{f}_+} R_{h(\lambda)} E(d\lambda)g$. To prove that $\widehat{h} \in D(A)$ we consider the integral $\int_{\mathbf{f}_+} |\lambda|^2 \langle E(d\lambda)\widehat{h}, \widehat{h}\rangle$. Since

$$\langle E(\alpha)\widehat{h}, \widehat{h}\rangle = \int\limits_{\mathbf{f}_+} \overline{h(\nu)} \langle E(\alpha)\widehat{h}, E(d\nu)g\rangle = \int\limits_{\mathbf{f}_+} \overline{h(\nu)} \langle \widehat{h}, E(\alpha)E(d\nu)g\rangle = \int\limits_{\alpha} \overline{h(\nu)} \langle \widehat{h}, E(d\nu)g\rangle$$

and

$$\langle \widehat{h}, E(\beta)g\rangle = \int\limits_{\mathbf{f}_+} \langle E(d\xi)g, E(\beta)g\rangle h(\xi) = \int\limits_{\beta} \langle E(d\xi)g, g\rangle h(\xi) = \int\limits_{\beta} \sigma(d\xi)h(\xi);$$

therefore,

$$\langle E(\alpha)\widehat{h}, \widehat{h}\rangle = \int\limits_{\alpha} |h(\nu)|^2 \sigma(d\nu).$$

Hence

$$\int\limits_{\mathbf{f}_+} |\lambda|^2 \langle E(d\lambda)\widehat{h}, \widehat{h}\rangle = \int\limits_{\mathbf{f}_+} |\lambda|^2 |h(\lambda)|^2 \sigma(d\lambda) = \int\limits_{[af,bf]} |\lambda|^2 |h(\lambda)|^2 \sigma(d\lambda) < \infty.$$

Further, for any vector $\widehat{h} = \Phi h \in \mathcal{D}(A) \quad A\widehat{h} = \int_{\mathbf{f}_+} R_{-\lambda f} J E(d\lambda)\widehat{h}$. Since

$$E(\alpha)\widehat{h} = \int\limits_{\mathbf{f}_+} E(\alpha) R_{h(nu)} E(d\nu)g = \int\limits_{\mathbf{f}_+} R_{h(\nu)} E(\alpha)E(d\nu)g = \int\limits_{\alpha} R_{h(\nu)} E(d\nu)g;$$

therefore, by linearity of $R_{-\lambda f}$ and J; and commutativity of J and E we have

$$A\widehat{h} = \int\limits_{\mathbf{f}_+} R_{-\lambda f} J R_{h(\lambda)} E(d\lambda)g = \int\limits_{\mathbf{f}_+} R_{h(\lambda)} R_{-\lambda f} E(d\lambda) Jg = \int\limits_{\mathbf{f}_+} R_{h(\lambda)} R_{-\lambda f} E(d\lambda) R_f g =$$

$$= \int\limits_{\mathbf{f}_+} R_{h(\lambda)} R_\lambda E(d\lambda)g = \int\limits_{\mathbf{f}_+} R_{\lambda h(\lambda)} E(d\lambda)g.$$

$\square$

Thus, on the set of finite functions from L^2_σ the equality $A\Phi = \Phi Q$ holds true (or, in other words, $A = \Phi Q \Phi^{-1}$). As a consequence we obtain the main result of this paper.

Theorem 3. *Any skew–selfadjoint operator with a simple spectrum acting on a Hilbert quaternion bimodule is unitarily equivalent to the operator of left multiplication by an independent variable on a functional bimodule* L^2_σ.

References

1. Viswanath K. *Normal operators on quaternionic Hilbert spaces* // Trans. Amer. Math. Soc. — 1971. — V. 162. — P. 337–350
2. Karpenko I. I., Suchtayev A. I., Tyshkevich D. L. *On one approach to differentiability of functions of a quaternionic variable* (In Russian) // Uchenye zapiski Tavricheskogo nacionalnogo universiteta imeni V. I. Vernadskogo, seria "Matematika. Mechanika. Informatika i Kibernetika." — 2004. — V.17(56), 1. — P. 30–37
3. Karpenko I. I., Tyshkevich D. L. *Spectral decomposition of a normal operator in a quatenionic Hilbert bimodule* // Book of abstracts, The Banach Center Conference Analysis and Partial Differential Equations (In honor of Professor Bogdan Bojarski), June 18–24, 2006, Mathematical Research and Conference Center, Poland, Będlewo. — 2006. — P. 21–22
4. Karpenko I. I., Suchtayev A. I., Tyshkevich D. L. *On the spectral decomposition of normal operators on a Hilbert quaternion bimodules* (In Russian) // Uchenye zapiski Tavricheskogo nacionalnogo universiteta imeni V. I. Vernadskogo, seria "Matematika. Mechanika. Informatika i Kibernetika." — 2006. — V.19(58), 1. — P. 3–20
5. Karpenko I. I., Tyshkevich D. L. *Spectral properties of linear operators on quaternionic bimodules* (In Russian) // Matematychni Studii. — 2008. — V. 30, 1. — P. 67–82